Math 53, Discussions 116 and 118 **2022-09-14**

Working with vectors

Questions

Question 1. Do the vectors $\langle 1, 2, 4\rangle$ and $\langle -2, -2, 2\rangle$ form an acute, right, or obtuse angle (when placed tail to tail)?

Question 2. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^3$, we have the two identities

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta, \qquad |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Which of these formulas is better for determining θ, and why?

Question 3. Find where the line through the points $(1, 2, 3)$ and $(2, 4, 6)$ intersects the plane $x + 3y + z = 20$.

Question 4. On a previous discussion worksheet, you investigated the equation

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$$

where $\mathbf{r} = \langle x, y\rangle$, $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ where a_1, a_2, b_1, b_2 are constants. After some tedious algebra, we found that this equation describes a circle such that the line segment connecting (a_1, a_2) and (b_1, b_2) is its diameter.

With your newfound knowledge (geometric understanding of the dot product), see if you can arrive at that same conclusion geometrically.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. We know that $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ where θ is the angle between the vectors $\mathbf{u}, \mathbf{v}$. If we are only interested in whether the angle is acute, right, or obtuse, it is sufficient to look at the sign of $\mathbf{u}\cdot\mathbf{v}$. In our case, $\langle 1, 2, 4\rangle \cdot \langle -2, -2, 2\rangle = -2-4+8 = 2 > 0$. Since $\cos\theta > 0$, it follows that $0 \le \theta < \pi/2$, i.e. that the angle is acute.

Question 2. The first equation is better (and not just because the dot product is easier to compute!). This is because $\cos\theta$ uniquely determines θ in the interval $[0, \pi]$, whereas $\sin\theta$ cannot distinguish between θ and $\pi - \theta$. Stated a different way, $\cos\theta$ is one-to-one on that interval, while $\sin\theta$ is not.

Question 3. The line can be parametrized as

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t(\langle 2, 4, 6\rangle - \langle 1, 2, 3\rangle).$$

After substituting into the equation $x + 3y + z = 20$ and solving for t, we get $t = 1$, which means that the point of intersection is $(2, 4, 6)$.